

# Groups with positive rank gradient and their actions

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## Abstract

We show that given a finitely generated LERF group  $G$  with positive rank gradient, and finitely generated subgroups  $A, B \leq G$  of infinite index, one can find a finite index subgroup  $B_0$  of  $B$  such that  $[G : \langle A \cup B_0 \rangle] = \infty$ . This generalizes a theorem of Olshanskii on free groups. We conclude that a finite product of finitely generated subgroups of infinite index does not cover  $G$ . We construct a transitive virtually faithful action of  $G$  such that the orbits of finitely generated subgroups of infinite index are finite. Some of the results extend to profinite groups with positive rank gradient.

## 1 Introduction

The rank gradient of a finitely generated group  $G$  is defined to be

$$\nabla G := \inf_U \frac{d(U) - 1}{[G : U]} \quad (1.1)$$

where  $U$  ranges over all subgroups of finite index in  $G$ , and  $d(U)$  stands for the smallest cardinality of a generating set of  $U$ . The notion of rank gradient has been introduced in [7] and further studied, for instance in [1], [2], [3], [6], [9], [10], and [11]. It is our point of view that many interesting properties of a group (e.g. a free group) can be deduced using only the positivity of its rank gradient, as explained and demonstrated in [13]. However, in order to effectively use the rank gradient, we inevitably need to make an additional assumption that will provide us with some finite index subgroups to which we can apply (1.1). This is achieved by considering the profinite topology of a group.

We think of all the groups as being topological by endowing them with the profinite topology, i.e. by taking as a basis the cosets of finite index subgroups. In this vein, recall that a group  $G$  is LERF (locally extended residually finite), or subgroup separable, if its finitely generated subgroups are closed, or equivalently, if each finitely generated subgroup of  $G$  is the intersection of some collection of finite index subgroups of  $G$ .

Our first result generalizes [8, Theorem 1.1] which is the statement one gets by taking  $G$  to be a nonabelian free group in the following.

**Theorem 1.1.** *Let  $G$  be a finitely generated LERF group with positive rank gradient, and let  $A, B$  be finitely generated subgroups of infinite index in  $G$ . Then there exists a finite index subgroup  $H$  of  $B$  such that  $A$  and  $H$  generate a subgroup of infinite index in  $G$ .*

As in [8, Theorem 1.1], if we are also given a finite subset  $S \subseteq G \setminus A$ , then the fact that  $G$  is LERF gives us a finite index subgroup  $U$  of  $G$  which contains  $A$  and avoids  $S$ . By taking  $H_0 := H \cap U$  we also assure that  $\langle A \cup H_0 \rangle$  avoids  $S$ . A consequence of Theorem 1.1 is:

**Theorem 1.2.** *Let  $G$  be a finitely generated LERF group with positive rank gradient, let  $n \in \mathbb{N}$ , and let  $H_1, \dots, H_n$  be finitely generated subgroups of infinite index in  $G$ . Then*

$$H_1 H_2 \cdots H_n \subsetneq G. \quad (1.2)$$

This means that  $G$  is not boundedly generated in a rather strong sense, thus improving upon [12] in the LERF case. Another application of Theorem 1.1 is the construction of 'locally finite' actions. Similar actions of free and hyperbolic groups are constructed in [8] and [4].

**Theorem 1.3.** *Let  $G$  be a finitely generated LERF group with  $\nabla G > 0$ . Then there exists a (right) transitive action of  $G$  on a set  $X$  such that:*

- *There are only finitely many  $g \in G$  which act trivially on  $X$ .*
- *For every finitely generated subgroup  $L$  of infinite index in  $G$ , and any  $x \in X$ , the orbit  $xL$  is finite.*

This gives us an almost faithful action on an infinite set which is 'locally finite' even though it is transitive. For instance, it follows that for all  $g \in G$  and  $x \in X$  the set  $\{xg^n : n \in \mathbb{Z}\}$  is finite.

Some examples of groups to which our theorems apply are:

1. Nonabelian free groups and nonabelian limit groups.
2. Surface groups and nonabelian Fuchsian groups.
3. Free products of finitely generated LERF groups of order  $> 2$ .
4. Free products of infinite finitely generated LERF groups amalgamating a finite subgroup.
5. Free products of finitely generated nonabelian Fuchsian groups with cyclic amalgamation.
6. Free products of infinite finitely generated nilpotent groups amalgamating a finite cyclic subgroup.
7. Fundamental groups of connected sums of compact hyperbolic 3-manifolds.
8. Finitely presented LERF groups with deficiency  $\geq 2$ .
9. Graph groups whose graph is disconnected and does not contain neither an induced path of length 3, nor an induced square.

Our first two results hold under an assumption weaker than LERF, namely **LPF**, introduced in [5, Definition 3.11]. We say that a group  $G$  is **LPF** if every finitely generated subgroup  $H$  of infinite index in  $G$  is contained in a subgroup  $U$  of arbitrarily large finite index in  $G$ , or equivalently, if the closure of  $H$  in  $G$  is of infinite index. Furthermore, as explained in 5, Theorem 1.1 and Theorem 1.2 have natural analogues for profinite groups, as all of their subgroups are closed by definition.

## 2 Profinite measure

Let  $G$  be a group, and let  $\mathcal{P}(G)$  be the family of its subsets. Define

$$\mu: \mathcal{P}(G) \rightarrow [0, 1], \quad \mu(S) := \inf_{\varphi} \frac{|\varphi(S)|}{|\varphi(G)|} \quad (2.1)$$

where  $\varphi$  ranges over all the epimorphisms from  $G$  onto finite groups. We call  $\mu$  the **profinite measure** on  $G$ , even though it is not a measure in case that  $G$  is infinite. The profinite measure does however enjoy the following properties, the trivial proof of which is omitted.

1. Monotonicity: for  $A \subseteq B \subseteq G$  we have

$$0 = \mu(\emptyset) \leq \mu(A) \leq \mu(B) \leq \mu(G) = 1. \quad (2.2)$$

2. Subadditivity: for  $A, B \subseteq G$  we have

$$\mu(A \cup B) \leq \mu(A) + \mu(B). \quad (2.3)$$

3. Translation invariance: for  $g, h \in G$ ,  $A \subseteq G$  we have

$$\mu(gAh) = \mu(A). \quad (2.4)$$

Since the profinite measure 'considers' only finite images, we see that its value on a set coincides with its value on the closure of the set. Also, if  $\widehat{G}$  is the profinite completion of  $G$ , then the profinite measure of a subset of  $G$ , is just the Haar measure of its closure in  $\widehat{G}$ . As we have already mentioned earlier, any closed subgroup  $H \leq G$  is the intersection of a family of finite index subgroups of  $G$ . In light of that, if  $[G : H] = \infty$  then  $H$  is contained in a subgroup of  $G$  with arbitrarily large finite index. We denote by  $H_G$  the normal core of  $H$  in  $G$ .

**Proposition 2.1.** *For a closed subgroup  $H$  of a group  $G$  we have*

$$\mu(H) = \frac{1}{[G : H]}. \quad (2.5)$$

*Proof.* First, suppose that  $n := [G : H] < \infty$ , and let  $\{g_1, \dots, g_n\}$  be a right transversal of  $H$  in  $G$ . The following inequalities give us (2.5):

$$\mu(H) \stackrel{2.1}{\leq} \frac{|H/H_G|}{|G/H_G|} = \frac{[H : H_G]}{[G : H_G]} = \frac{[H : H_G]}{[G : H][H : H_G]} = \frac{1}{n}. \quad (2.6)$$

$$1 \stackrel{2.2}{=} \mu(G) = \mu\left(\bigcup_{i=1}^n Hg_i\right) \stackrel{2.3}{\leq} \sum_{i=1}^n \mu(Hg_i) \stackrel{2.4}{=} \sum_{i=1}^n \mu(H) = n\mu(H). \quad (2.7)$$

Now, suppose that  $[G : H] = \infty$ , and take some  $\epsilon > 0$ . Since  $H$  is closed, there exists some  $H \leq U \leq G$  with  $\frac{1}{\epsilon} \leq [G : U] < \infty$ . Hence,

$$\mu(H) \stackrel{2.2}{\leq} \mu(U) = \frac{1}{[G : U]} \leq \epsilon. \quad (2.8)$$

□

### 3 Finitely generated subgroups

Following [5], we define the **proindex** of a subgroup  $H$  in a group  $G$  to be the supremum of indices of finite index subgroups of  $G$  above  $H$ .

**Lemma 3.1.** *Let  $G$  be a finitely generated group with  $\nabla G > 0$ , and let  $A, B$  be finitely generated subgroups of infinite proindex in  $G$ . Then there exist finite index subgroups  $A_0 \leq A, B_0 \leq B$  which generate a subgroup of infinite index in  $G$ .*

*Proof.* Since the proindices are infinite, there exist finite index subgroups  $U, V \leq G$  containing  $A, B$  respectively, such that

$$[G : U], [G : V] \geq \frac{2 \max\{d(A), d(B)\}}{\nabla G}. \quad (3.1)$$

Set  $A_0 := A \cap V, B_0 := B \cap U, C := \langle A_0 \cup B_0 \rangle$ , and note that

$$\begin{aligned} [A : A_0] &= [A : A \cap U \cap V] \leq [U : U \cap V] \\ [B : B_0] &= [B : B \cap U \cap V] \leq [V : U \cap V]. \end{aligned} \quad (3.2)$$

Using Schreier's bound on the rank of a finite index subgroup, we get

$$\begin{aligned} d(C) &\leq d(A_0) + d(B_0) \leq [A : A_0]d(A) + [B : B_0]d(B) \\ &\stackrel{3.2}{\leq} [U : U \cap V]d(A) + [V : U \cap V]d(B) \\ &= [G : U \cap V] \left( \frac{d(A)}{[G : U]} + \frac{d(B)}{[G : V]} \right) \\ &\leq \frac{2[G : U \cap V] \max\{d(A), d(B)\}}{\min\{[G : U], [G : V]\}} \\ &\stackrel{3.1}{\leq} [G : U \cap V] \nabla G. \end{aligned} \quad (3.3)$$

Since  $A_0, B_0 \leq U \cap V$ , we see that  $C \leq U \cap V$ . Were the index  $[G : C]$  finite, we would have the following contradiction to (3.3):

$$d(C) \stackrel{1.1}{\geq} [G : C] \nabla G + 1 \geq [G : U \cap V] \nabla G + 1. \quad (3.4)$$

□

As in [5, Definition 3.11], we say that a group  $G$  is **LPF** if the index and proindex coincide for every finitely generated subgroup of  $G$ .

**Corollary 3.2.** *Let  $G$  be a finitely generated LPF group with  $\nabla G > 0$ , and let  $A, B$  be finitely generated subgroups of infinite index in  $G$ . Then*

$$\mu(AB) = 0. \quad (3.5)$$

*Proof.* By Lemma 3.1, there exist finite index subgroups  $A_0, B_0$  of  $A, B$  respectively, such that the index of  $C := \langle A_0 \cup B_0 \rangle$  in  $G$  is infinite. Since  $A_0$  and  $B_0$  are finitely generated,  $C$  is finitely generated as well, so its proindex in  $G$  is infinite as  $G$  is LPF. Therefore, by Proposition 2.1,

$$\mu(C) = 0. \quad (3.6)$$

Taking  $L$  to be a left transversal of  $A_0$  in  $A$ , and  $R$  to be a right transversal of  $B_0$  in  $B$ , we get

$$\begin{aligned} \mu(AB) &= \mu\left(\bigcup_{L,R} \ell A_0 B_0 r\right) \stackrel{2.3}{\leq} \sum_{L,R} \mu(\ell A_0 B_0 r) \stackrel{2.4}{=} \sum_{L,R} \mu(A_0 B_0) \\ &= |L||R|\mu(A_0 B_0) \stackrel{2.2}{\leq} |L||R|\mu(C) \stackrel{3.6}{=} 0. \end{aligned} \quad (3.7)$$

□

We need a simple observation for the proof of Theorem 1.1.

**Proposition 3.3.** *Let  $\varphi: G \rightarrow K$  be a group homomorphism, let  $N$  be its kernel, and let  $A, B \subseteq G$ . Then  $\varphi(A) \cap \varphi(B) \subseteq \varphi(B \cap NA)$ .*

*Proof.* Let  $z \in \varphi(A) \cap \varphi(B)$ . There exist  $a \in A$ ,  $b \in B$  such that  $z = \varphi(a) = \varphi(b)$ . Thus,  $b \in Na \subseteq NA$ , so  $z = \varphi(b) \in \varphi(B \cap NA)$ . □

Let us now prove Theorem 1.1.

**Theorem 3.4.** *Let  $G$  be a finitely generated LPF group with  $\nabla G > 0$ , and let  $A, B$  be finitely generated subgroups of infinite index in  $G$ . Then there is a finite index subgroup  $B_0 \leq B$  such that  $[G : \langle A \cup B_0 \rangle] = \infty$ .*

*Proof.* Set

$$r := \max\{d(A), d(B)\}, \quad \epsilon := \frac{\nabla G}{2r}. \quad (3.8)$$

By Corollary 3.2,  $\mu(AB) = 0$ , so there exists an epimorphism onto a finite group  $\varphi: G \rightarrow K$ , such that

$$\frac{|\varphi(AB)|}{|K|} \stackrel{2.1}{\leq} \epsilon. \quad (3.9)$$

Put

$$N := \text{Ker}(\varphi), \quad B_0 := B \cap NA, \quad C := \langle A \cup B_0 \rangle \quad (3.10)$$

and note that  $B \cap N = B_0 \cap N$ , so

$$\begin{aligned}
[B : B_0] &= \frac{[B : B \cap N]}{[B_0 : B \cap N]} = \frac{[B : B \cap N]}{[B_0 : B_0 \cap N]} \stackrel{3.10}{=} \frac{|\varphi(B)|}{|\varphi(B_0)|} \\
&\stackrel{3.3}{\leq} \frac{|\varphi(B)|}{|\varphi(A) \cap \varphi(B)|} = \frac{|\varphi(A)\varphi(B)|}{|\varphi(A)|} = \frac{|\varphi(AB)|}{|\varphi(A)|} \\
&\stackrel{3.9}{\leq} \frac{\epsilon|K|}{|\varphi(A)|} = \epsilon[K : \varphi(A)] \stackrel{3.10}{=} \epsilon[G : NA].
\end{aligned} \tag{3.11}$$

Applying Schreier's bound we see that

$$\begin{aligned}
d(C) &\stackrel{3.10}{\leq} d(A) + d(B_0) \leq d(A) + [B : B_0]d(B) \\
&\stackrel{3.8}{\leq} r(1 + [B : B_0]) \leq 2r[B : B_0] \stackrel{3.11}{\leq} 2r\epsilon[G : NA] \\
&\stackrel{3.8}{=} [G : NA]\nabla G.
\end{aligned} \tag{3.12}$$

If  $[G : C]$  were finite, we would get a contradiction to (3.12):

$$d(C) \stackrel{1.1}{\geq} [G : C]\nabla G + 1 \stackrel{3.10}{\geq} [G : NA]\nabla G + 1. \tag{3.13}$$

□

## 4 Corollaries

We give some corollaries of Theorem 3.4, the first of which is a strong form of Theorem 1.2.

**Corollary 4.1.** *Let  $G$  be a finitely generated LPF group with  $\nabla G > 0$ , let  $n \in \mathbb{N}$ , and let  $H_1, \dots, H_n$  be finitely generated subgroups of infinite index in  $G$ . Then*

$$\mu(H_1 \cdots H_n) = 0. \tag{4.1}$$

*Proof.* We induct on  $n$ , using Proposition 2.1 to see that the base case  $n = 1$  holds. For  $n \geq 2$ , Theorem 3.4 gives us a finitely generated subgroup  $C$  of infinite index in  $G$  which contains both  $H_{n-1}$  and a finite index subgroup  $H$  of  $H_n$ . By induction,

$$\mu(H_1 \cdots H_{n-2}C) = 0 \tag{4.2}$$

so by taking  $R$  to be a right transversal of  $H$  in  $H_n$ , we see that

$$\begin{aligned}
\mu(H_1 \cdots H_{n-2}H_{n-1}H_n) &= \mu(H_1 \cdots H_{n-2}H_{n-1}HR) \\
&\stackrel{2.2}{\leq} \mu(H_1 \cdots H_{n-2}\langle H_{n-1} \cup H \rangle R) \\
&\stackrel{2.2}{\leq} \mu(H_1 \cdots H_{n-2}CR) \\
&= \mu\left(\bigcup_{r \in R} H_1 \cdots H_{n-2}Cr\right) \\
&\stackrel{2.3}{\leq} \sum_{r \in R} \mu(H_1 \cdots H_{n-2}Cr) \\
&\stackrel{2.4}{=} |R|\mu(H_1 \cdots H_{n-2}C) \stackrel{4.2}{=} 0.
\end{aligned} \tag{4.3}$$

□

In order to prove Theorem 1.3, we follow the argument in the proof of [8, Corollary 4.1 (a)], and instead of using [8, Theorem 1.1], we invoke Theorem 3.4. The conclusion is that any LPF group  $G$  with  $\nabla G > 0$  contains an infinite index subgroup  $R$  such that every finitely generated subgroup  $L$  of infinite index in  $G$  contains  $L \cap R$  as a finite index subgroup. In other words,  $G$  acts transitively on  $X := R \backslash G$  such that for every  $x \in X$  and every finitely generated subgroup  $L$  of infinite index in  $G$ , the orbit  $xL$  is finite (see the proof of [8, Corollary 4.5 (a)] for a detailed explanation). Hence, in order to establish Theorem 1.3, we only need to show that the kernel of the action (those  $g$  in  $G$  which act trivially) is finite. For that, recall that  $G$  is said to be **RF** (residually finite) if for every  $K \subseteq G$  with  $|K| \geq M \in \mathbb{R}$ , there exists a finite index subgroup  $U \triangleleft G$  such that  $|KU/U| \geq M$ . If  $G$  is LERF, it is also RF.

**Theorem 4.2.** *Let  $G$  be a finitely generated residually finite group with  $\nabla G > 0$ , and let  $R$  be an infinite index subgroup of  $G$  such that for every finitely generated subgroup  $L$  of infinite index in  $G$  we have  $[L : L \cap R] < \infty$ . Then  $K := \text{Ker}(R \backslash G \curvearrowright G)$  is finite.*

*Proof.* Towards a contradiction, suppose that  $|K| \geq \frac{d(G)}{\nabla G}$ . Since  $G$  is residually finite, there exists a finite index subgroup  $U \triangleleft G$  such that

$$|KU/U| \geq \frac{d(G)}{\nabla G}. \quad (4.4)$$

For every finite index subgroup  $V \leq U$  we have

$$\begin{aligned} d(U/U \cap K) &= d(UK/K) \leq d(UK) \leq d(G)[G : UK] \\ &= \frac{d(G)[G : U]}{[UK : U]} \leq \frac{d(G)[G : V]}{[KV : U]} \\ &\stackrel{1.1}{\leq} \frac{d(G)(d(V) - 1)}{\nabla G[KU : U]} \stackrel{4.4}{\leq} d(V) - 1. \end{aligned} \quad (4.5)$$

Let  $T$  be a generating set of  $U/U \cap K$  of minimal cardinality, let  $S \subseteq U$  be a set mapped bijectively to  $T$  by the quotient map  $q : U \rightarrow U/U \cap K$ , and set  $L := \langle S \rangle$ . Clearly,  $L \leq U$  and  $L(U \cap K) = U$  since  $T \subseteq q(L)$ . Were the index of  $L$  in  $U$  finite, we would get a contradiction as follows:

$$d(L) \leq |S| = |T| = d(U/U \cap K) \stackrel{4.5}{\leq} d(L) - 1. \quad (4.6)$$

Hence,  $[U : L] = \infty$ . By our assumption, there exists a finite left transversal  $F$  of  $L \cap R$  in  $L$ . As  $K \leq R$ , it follows that

$$U = L(U \cap K) \subseteq L(U \cap R) = F(L \cap R)(U \cap R) \subseteq F(U \cap R) \quad (4.7)$$

so  $[U : U \cap R] < \infty$  and thus  $[G : R] = \frac{[G : U][U : U \cap R]}{[R : U \cap R]} < \infty$  - an absurdity. □

## 5 Profinite groups

Since our arguments are focused mainly on finite index subgroups, some of our results are more naturally stated for profinite groups. For these groups, only closed subgroups are considered, so additional separability assumptions such as RF, LPF, LERF are not required.

**Theorem 5.1.** *Let  $\Gamma$  be a finitely generated profinite group with positive rank gradient, and let  $A, B$  be finitely generated subgroups of infinite index in  $\Gamma$ . Then there exists an open subgroup  $B_0$  of  $B$  such that  $A$  and  $B_0$  generate a subgroup of infinite index in  $\Gamma$ .*

The proof is identical to that of Theorem 3.4.

**Theorem 5.2.** *Let  $\Gamma$  be a finitely generated profinite group with positive rank gradient, let  $n \in \mathbb{N}$ , and let  $H_1, \dots, H_n$  be finitely generated subgroups of infinite index in  $\Gamma$ . Then*

$$\mu_\Gamma(H_1 \cdots H_n) = 0. \quad (5.1)$$

Here  $\mu_\Gamma$  stands for the Haar measure on  $\Gamma$ . The proof follows that of Corollary 4.1. Some profinite groups with positive rank gradient are:

1. Nonabelian free profinite, free pro- $p$ , and free prosolvable groups.
2. Free pro- $p$  products.
3. Groups satisfying Schreier's formula.
4. Nonsolvable Demushkin groups and surface groups.
5. Pro- $p$  groups with deficiency at least 2.
6. Pro- $p$  groups from the class  $\mathcal{L}$  all of whose abelian subgroups are procyclic.
7. Completions of groups from the list in the introduction.

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